*Claim to be proven:* For any complete, consistent, saturated set of structure Γ, we have ∀xA(x) ∈ Γ if, and only if, for each closed term *t*, we have A(t) ∈ Γ.

*Proof:*

For the left-to-right direction (the “if” direction), we suppose that ∀xA(x) ∈ Γ, for conditional introduction.   
  
By proposition 9.26, and by 9.14 (Monotony), we know that, for any closed term *t*, we have Γ A(t).

For contradiction, suppose A(t) ∈ Γ. Since Γ is complete, we can infer that ¬A(t) ∈ Γ. Via proposition 9.20, we can now infer that Γ ⊢ ⊥. But this contradicts that Γ is consistent. Because ¬A(t) ∈ Γ leads to a contradiction, we can derive A(t) ∈ Γ. Since *t* was an arbitrary, closed term, we have completed the proof for the left-to-right (if) direction.

For the right-to-left direction (the “only if” direction), we suppose that, for every closed term *t*, we have A(t) ∈ Γ.   
  
For contradiction, suppose ∀xA(x) ~~∈~~ Γ. Since Γ is complete, by the definition of completeness, we know that ¬∀xA(x) ∈ Γ, and therefore can conclude that Γ ¬∀xA(x). From this, we can conclude that Γ ∃x¬A(x).  
  
To understand how this chain of reasoning follows, consider the following derivation. Suppose, for negation introduction, A(a) for a constant *a* that is not in A. We can now apply ∀ introduction to derive ∀xA(x). This contradicts ¬∀xA(x), which we derived from the starting point of this direction of the proof. We can thus derive ⊥ by negation elimination, and from ⊥, we can derive ~A(a) by negation introduction, discharging our assumption of A(a). From ~A(a), we can apply an existential introduction, and arrive at Ex~A(x).   
  
Note that, since Γ is saturated, by the definition of saturation (10.4), it must be the case that ~A(x) exists in the enumeration of all formulas that have one free variable ~A(x) =An (xn) for some *n* , using definition 10.5. If so, also by 10.4, we have Dn = ∃xn ~A(xn ) → ~A(cn) ∈ Γ. By 9.24, we therefore have ~A(cn ) ∈ Γ. But given that cn is a closed term, from the assumption at the start of the “only if” direction, that A(t) ∈ Γ, we also have that ¬A(cn ) ∈ Γ. This contradicts that Γ was consistent. Therefore, the original assumption, ∀xA(x) ~~∈~~ Γ, must have been false, and we can conclude that ∀xA(x) ∈ Γ.

This amounts to a proof of the right-to-left direction, and the completion of the overall proof.

~~∈~~ is meant to be the struck-out ∈, but Word has trouble with that symbol in particular.

**Your two directions should be switched (i.e., your first subproof is actually the only if direction and your second subproof is actually the if direction.) This was an error on the Carnap sheet itself, so on a test you would not be marked down for this. However, what you would be marked for your first subproof (the only if direction): it does not establish the claim.**

**It's good that you start from the assumption that ∀x A(x) ∈ Γ. Now you should create a derivation to complete this direction. Hint: your derivation should start with ∀x A. After your derivation, you'll need to appeal to some of your assumptions to show why A(t) ∈ Γ. (Also, please see page 182 of the text for the definition of monotony. You won't need it for this proof, but I thought I should mention this to you.)**